

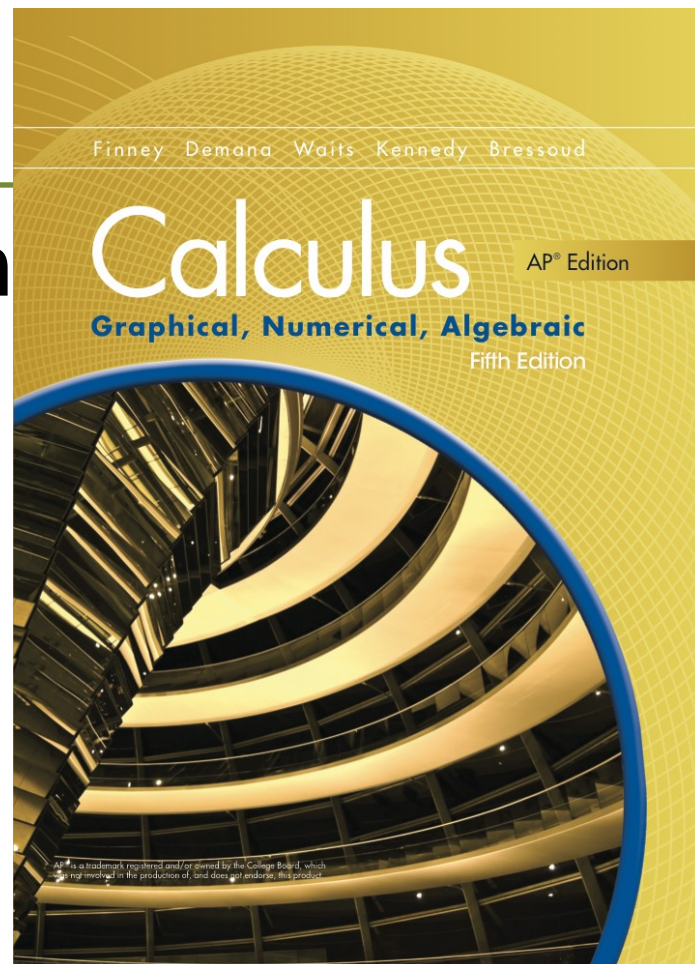
Chapter 3

Derivatives



Section 3.1

Derivative of a Function



Quick Review

In Exercises 1 – 4, evaluate the indicated limit algebraically

1. $\lim_{h \rightarrow 0} \frac{(2+h)^2 - 4}{h}$

2. $\lim_{x \rightarrow 2^+} \frac{x+3}{2}$

3. $\lim_{y \rightarrow 0} \frac{|y|}{y}$

4. $\lim_{x \rightarrow 4} \frac{2x-8}{\sqrt{x}-2}$

5. Find the slope of the line tangent to the parabola $y = x^2 + 1$ at its vertex.

6. By considering the graph of $f(x) = x^3 - 3x^2 + 2$, find the intervals on which f is increasing.

Quick Review

In Exercises 7 – 10, let

$$f(x) = \begin{cases} x + 2, & x \leq 1 \\ (x - 1)^2, & x > 1 \end{cases}$$

7. Find $\lim_{x \rightarrow 1^+} f(x)$ and $\lim_{x \rightarrow 1^-} f(x)$.
8. Find $\lim_{h \rightarrow 0^+} f(1 + h)$.
9. Does $\lim_{x \rightarrow 1} f(x)$ exist? Explain.
10. Is f continuous? Explain.

Quick Review Solutions

In Exercises 1- 4, evaluate the indicated limit algebraically

1. $\lim_{h \rightarrow 0} \frac{(2+h)^2 - 4}{h}$ 4

2. $\lim_{x \rightarrow 2^+} \frac{x+3}{2}$ $\frac{5}{2}$

3. $\lim_{y \rightarrow 0} \frac{|y|}{y}$ - 1

4. $\lim_{x \rightarrow 4} \frac{2x-8}{\sqrt{x}-2}$ 8

5. Find the slope of the line tangent to the parabola $y=x^2 + 1$ at its vertex. 0

6. By considering the graph of $f(x) = x^3 - 3x^2 + 2$, find the intervals on which f is increasing. $(-\infty, 0]$ and $[2, \infty)$

Quick Review Solutions

In Exercises 7 – 10, let

$$f(x) = \begin{cases} x + 2, & x \leq 1 \\ (x - 1)^2, & x > 1 \end{cases}$$

7. Find $\lim_{x \rightarrow 1^+} f(x)$ and $\lim_{x \rightarrow 1^-} f(x)$. **0; 3**

8. Find $\lim_{h \rightarrow 0^+} f(1 + h)$. **0**

9. Does $\lim_{x \rightarrow 1} f(x)$ exist? Explain. **No.**

The two one-sided limits are different.

10. Is f continuous? Explain. **No, f is discontinuous at $x=1$ because the limit does not exist there.**

What you'll learn about

- The meaning of differentiable
- Different ways of denoting the derivative of a function
- Graphing $y = f(x)$ given the graph of $y = f'(x)$
- Graphing $y = f'(x)$ given the graph of $y = f(x)$
- One-sided derivatives
- Graphing the derivative from data

... and why

The derivative gives the value of the slope of the tangent line to a curve at a point.

Definition of Derivative

The **derivative** of the function f with respect to the variable x is the function $f'(x)$ whose value at x is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

Differentiable Function

The domain of f' , the set of points in the domain of f for which the limit exists, may be smaller than the domain of f . If $f'(x)$ exists, we say that **f has a derivative (is differentiable) at x** . A function that is differentiable at every point in its domain is a **differentiable function**.

Example Definition of Derivative

Differentiate $f(x) = x^2$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

substitute

$$= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2) - x^2}{h}$$

$(x+h)^2$ expanded

$$= \lim_{h \rightarrow 0} \frac{(2x+h)h}{h} \quad x^2$$

cancelled and h factored out

$$= \lim_{h \rightarrow 0} (2x+h)$$

$$= 2x$$

Derivative at a Point (alternate)

The derivative of the function f at the point where $x=a$ is the limit

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists.

Notation

There are many ways to denote the derivative of a function $y = f(x)$.

Besides $f'(x)$, the most common notations are:

y'	“ y prime”	Nice and brief, but does not name the independent variable.
$\frac{dy}{dx}$	“ $dy\ dx$ ” or “the derivative of y with respect to x ”	Names both variables and uses d for derivative.
$\frac{df}{dx}$	“ $df\ dx$ ” or “the derivative of f with respect to x ”	Emphasizes the function’s name.
$\frac{d}{dx} f(x)$	“ $d\ dx$ of f at x ” or “the derivative of f at x ”	Emphasis that differentiation is an operation performed on f .

Relationships between the Graphs of f and f'

Because we can think of the derivative at a point in graphical terms as slope, we can get a good idea of what the graph of the function f' looks like by *estimating the slopes* at various points along the graph of f .

We estimate the slope of the graph of f in y -units per x -unit at frequent intervals. We then plot the estimates in a coordinate plane with the horizontal axis in x -units and the vertical axis in slope units.

Graphing the Derivative from Data

Discrete points plotted from sets of data do not yield a continuous curve, but we have seen that the shape and pattern of the graphed points (called a scatter plot) can be meaningful nonetheless. It is often possible to fit a curve to the points using regression techniques. If the fit is good, we could use the curve to get a graph of the derivative visually. However, it is also possible to get a scatter plot of the derivative numerically, directly from the data, by computing the slopes between successive points.

One-sided Derivatives

A function $y = f(x)$ is **differentiable on a closed interval $[a, b]$** if it has a derivative at every interior point on the interval, and if the limits

$$\lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h} \quad \left[\text{the **right-hand derivative at } a \right]**$$

$$\lim_{h \rightarrow 0^-} \frac{f(b+h) - f(b)}{h} \quad \left[\text{the **left-hand derivative at } b \right]**$$

exist at the endpoints. In the right-hand derivative, h is positive and $a+h$ approaches a from the right. In the left-hand derivative, h is negative and $b+h$ approaches b from the left.

One-sided Derivatives

Right-hand and left-hand derivatives may be defined at any point of a function's domain.

The usual relationship between one-sided and two-sided limits holds for derivatives. Theorem 3, Section 2.1, allows us to conclude that a function has a (two-sided) derivative at a point if and only if the function's right-hand and left-hand derivatives are defined and equal at that point.

Example One-sided Derivatives

Show that the following function has left-hand and right-hand derivatives at $x=0$, but no derivative there.

$$y = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

Left-hand derivative:

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{-(0+h) - 0}{h} \\ = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1 \end{aligned}$$

Right-hand derivative:

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{(0+h) - 0}{h} \\ = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1 \end{aligned}$$

The derivatives are not equal at $x=0$. The function does not have a derivative at 0.